# Atomic Cournotian Traders May Be Walrasian*

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**Abstract**

In a bilateral oligopoly, with large traders, represented as atoms, and small traders, represented by an atomless part, when is there a non-empty intersection between the sets of Walras and Cournot-Nash allocations? Using a two-commodity version of the Shapley window model, we show that a necessary and sufficient condition for a Cournot-Nash allocation to be a Walras allocation is that all atoms demand a null amount of one of the two commodities. We provide four examples which show that this characterization holds non-vacuously. When our condition fails to hold, we also confirm, through some examples, the result obtained by Okuno et al. (1980): small traders always have a negligible influence on prices, while the large traders keep their strategic power even when their behavior turns out to be Walrasian in the cooperative framework considered by Gabszewicz and Mertens (1971) and Shitovitz (1973).

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## 1 Introduction

In his celebrated paper, Aumann (1964) proved that, in exchange economies with a continuum of traders, the core coincides with the set of Walras al-

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locations. Some years later, Gabszewicz and Mertens (1971) and Shitovitz (1973) introduced the notion of a mixed exchange economy, i.e., an exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part, in order to analyze oligopoly in a general equilibrium framework. Gabszewicz and Mertens (1971) showed that, if atoms are not "too" big, the core still coincides with the set of Walras allocations whereas Shitovitz (1973), in his Theorem B, proved that this result also holds if the atoms are of the same type, i.e., have the same endowments and preferences.

Okuno et al. (1980) considered the result obtained by Shitovitz (1973) so counterintuitive to call into question the use of the core as the solution concept to study oligopoly in general equilibrium.[1] This led them to replace the core with the Cournot-Nash equilibrium of a model of simultaneous, noncooperative exchange between large traders and small traders as the appropriate solution for the analysis of oligopoly in general equilibrium. The model of noncooperative exchange they used belongs to a line of research initiated by Lloyd S. Shapley and Martin Shubik (see Giraud (2003) for a survey of this literature). In particular, they considered a mixed exchange economy with two commodities which are both held by all traders. Moreover, they assumed that no trader is allowed to be both buyer and seller of any commodity. In this framework, they showed that, if there are two atoms of the same type who demand, at a Cournot-Nash equilibrium, a positive amount of the two commodities, then the corresponding Cournot-Nash allocation is not a Walras allocation. Therefore, under the assumptions of Shitovitz's Theorem B, demanding a non-null amount of the two commodities by all the atoms is a sufficient condition for a Cournot-Nash allocation not to be a Walras allocation. This proposition allowed Okuno et al. (1980) to conclude that the noncooperative model they considered is a useful one to study oligopoly in a general equilibrium framework as the small traders always have a negligible influence on prices, while the large traders keep their strategic power even when their behavior turns out to be Walrasian in the cooperative framework considered by Shitovitz (1973).

In this paper, we raise the question whether, in mixed exchange economies, an equivalence, or at least a nonempty intersection, between the sets of Walras and Cournot-Nash allocations may hold. In order to further simplify our analysis, we consider the model of bilateral oligopoly introduced by

[1]Okuno et al. (1980) did not quote the result obtained by Gabszewicz and Mertens (1971). Nevertheless, their argument also holds, *mutatis mutandis*, for this result.

Gabszewicz and Michel (1997) and further analyzed by Bloch and Ghosal (1997), Bloch and Ferrer (2001), Dickson and Hartley (2008), Amir and Bloch (2009), among others. By using this model, we still remain in a two-commodity setting but we assume that each trader holds only one of the two commodities whose aggregate amount is strictly positive in the economy. In particular, we shall use a bilateral oligopoly version of the Shapley window model. This model was first proposed informally by Lloyd S. Shapley and further analyzed, in the case of finite economies, by Sahi and Yao (1989), in economies with an atomless continuum of traders, by Codognato and Ghosal (2000), and, in mixed exchange economies, by Busetto et al. (2011). In particular, Codognato and Ghosal (2000) proved that the sets of Walras and Cournot-Nash allocations coincide in economies with an atomless continuum of traders, thereby providing a noncooperative version of Aumann's theorem. Here, we first show, through some examples, that this threefold equivalence may not hold, in the bilateral oligopoly configuration, even under the assumptions made by Gabszewicz and Mertens (1971) and Shitovitz (1973), thereby confirming the result obtained by Okuno et al. (1980). We then answer our main question by proving a theorem which states that demanding a null amount of one of the two commodities by all the atoms is a necessary and sufficient condition for a Cournot-Nash allocation to be a Walras allocation. We also provide four examples which show that this characterization theorem is non-vacuous. Our result depends only on atoms' demand behavior at a Cournot-Nash equilibrium. This opens the door to a research on the conditions on the primitives of the model, i.e., traders' size, endowments, and preferences, under which our theorem holds. We start an investigation in this direction by providing a necessary condition, expressed in terms of bounds on atoms' marginal rates of substitution, for our result to hold when atoms' preferences are represented by additively separable utility functions as is the case for all the examples considered in this paper.

The paper is organized as follows. In Section 2, we introduce the mathematical model. In Section 3, we state the main equivalence theorems. In Section 4, we provide some examples and we state and prove our main theorem together with two propositions. In Section 5, we draw some conclusions from our analysis.[2]

[2]In the online appendix, we prove that our main theorem can be extended, in bilateral oligopoly, to other models of noncooperative exchange.

## 2 The mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where $T$ is the set of traders, $\mathcal{T}$ is the $\sigma$-algebra of all $\mu$-measurable subsets of $T$, and $\mu$ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. This implies that the measure space $(T, \mathcal{T}, \mu)$ contains at most countably many atoms. Let $T_0$ denote the atomless part of $T$. A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. A coalition is a nonnull element of $\mathcal{T}$. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are 2 different commodities. A commodity bundle is a point in $R^2_+$. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^2_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

**Assumption 1.** *There is a coalition $S$ such that $\mathbf{w}^1(t) > 0$, $\mathbf{w}^2(t) = 0$, for each $t \in S$, and $\mathbf{w}^1(t) = 0$, $\mathbf{w}^2(t) > 0$, for each $t \in S^c$.*

An allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^2_+ \to R$, satisfying the following assumptions.

**Assumption 2.** *$u_t : R^2_+ \to R$ is continuous, strongly monotone, and quasi-concave, for each $t \in T$.*

Let $\mathcal{B}(R^2_+)$ denote the Borel $\sigma$-algebra of $R^2_+$. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the $\sigma$-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

**Assumption 3.** *$u : T \times R^2_+ \to R$, given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R^2_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.*

An allocation $\mathbf{y}$ dominates an allocation $\mathbf{x}$ via a coalition $S$ if $u_t(\mathbf{y}(t)) \geq u_t(\mathbf{x}(t))$, for each $t \in S$, $u_t(\mathbf{y}(t)) > u_t(\mathbf{x}(t))$, for a nonnull subset of traders $t$ in $S$, and $\int_S \mathbf{y}(t)\, d\mu = \int_S \mathbf{w}(t)\, d\mu$. The core is the set of all allocations which are not dominated via any coalition.

A price vector is a nonnull vector $p \in R^2_+$. A Walras equilibrium is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector $p^*$ and an allocation $\mathbf{x}^*$, such that $p^*\mathbf{x}^*(t) = p^*\mathbf{w}(t)$ and $u_t(\mathbf{x}^*(t)) \geq u_t(y)$, for all $y \in \{x \in R^2_+ : p^*x = p^*\mathbf{w}(t)\}$, for each $t \in T$. A Walras allocation is an allocation $\mathbf{x}^*$ for which there exists a price vector $p^*$ such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium.

We now introduce the strategic market game considering the two-commodity version of the reformulation of the Shapley window model proposed by Busetto et al. (2011). A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R^4_+)$ such that, for each $t \in T$, $\mathbf{B}(t) = \{b \in R^4_+ : \sum_{j=1}^2 b_{ij} \leq \mathbf{w}^i(t),\ i = 1, 2\}$, where $b_{ij}$ represents the amount of commodity $i$ that trader $t$ offers in exchange for commodity $j$. A strategy selection is an integrable function $\mathbf{b} : T \to R^4_+$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. Given a strategy selection $\mathbf{b}$, we define the aggregate matrix $\overline{\mathbf{B}} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ the strategy selection obtained from $\mathbf{b}$ by replacing $\mathbf{b}(t)$ with $b(t) \in \mathbf{B}(t)$ and by $\overline{\mathbf{B}} \setminus b(t)$ the corresponding aggregate matrix.

We then introduce two further definitions (see Sahi and Yao (1989)).

**Definition 1.** *A nonnegative square matrix $A$ is said to be irreducible if, for every pair $(i, j)$, with $i \neq j$, there is a positive integer $k$ such that $a^{(k)}_{ij} > 0$, where $a^{(k)}_{ij}$ denotes the $ij$-th entry of the $k$-th power $A^k$ of $A$.*

**Definition 2.** *Given a strategy selection* $\mathbf{b}$, *a price vector $p$ is said to be market clearing if*

$$p \in R^2_{++}, \sum_{i=1}^{2} p^i \overline{\mathbf{b}}_{ij} = p^j (\sum_{i=1}^{2} \overline{\mathbf{b}}_{ji}),\ j = 1, 2. \tag{1}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector $p$ satisfying (1) if and only if $\overline{\mathbf{B}}$ is irreducible. Then, we denote by $p(\mathbf{b})$ a function which associates with each strategy selection $\mathbf{b}$ the unique, up to a scalar multiple, price vector $p$ satisfying (1), if $\overline{\mathbf{B}}$ is irreducible, and is equal to 0, otherwise.

Given a strategy selection $\mathbf{b}$ and a price vector $p$, consider the assignment determined as follows:

$$\begin{aligned}
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^{2} \mathbf{b}_{ji}(t) + \sum_{i=1}^{2} \mathbf{b}_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in R^2_{++}, \\
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise,}
\end{aligned}$$

$j = 1, 2$, for each $t \in T$.

Given a strategy selection $\mathbf{b}$ and the function $p(\mathbf{b})$, the traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),$$

for each $t \in T$.[3] It is straightforward to show that this assignment is an allocation.

We are now able to define a notion of Cournot-Nash equilibrium for this reformulation of the Shapley window model (see Codognato and Ghosal (2000) and Busetto et al. (2011)).

**Definition 3.** *A strategy selection* $\hat{\mathbf{b}}$ *such that* $\overline{\hat{\mathbf{B}}}$ *is irreducible is a Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

*for each* $b(t) \in \mathbf{B}(t)$ *and for each* $t \in T$.[4]

A Cournot-Nash allocation is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, where $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium.

# 3 The equivalence theorems

The following theorem reminds us that, when the space of traders is atomless, the core coincides with the set of Walras allocations as proved by Aumann (1964) which, in turn, coincides with the set Cournot-Nash allocations of the Shapley window model as shown by Codognato and Ghosal (2000).

**Theorem 1** [Aumann (1964), Codognato and Ghosal (2000)]**.** *Under Assumptions 1, 2, and 3, if* $T = T_0$*, then the core coincides with the sets of Walras and Cournot-Nash allocations.*

Gabszewicz and Mertens (1971) and Shitovitz (1973) showed that an equivalence between the core and the set of Walras allocations may hold even when the space of traders contains atoms. In order to state their two main theorems, we need to introduce some further notation and definitions.

[3]In order to save in notation, with some abuse, we denote by $\mathbf{x}$ both the function $\mathbf{x}(t)$ and the function $\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b}))$.

[4]Let us notice that, as this definition of a Cournot-Nash equilibrium explicitly refers to irreducible matrices, it applies only to active Cournot-Nash equilibria (on this point, see Sahi and Yao (1989)).

Two traders $\tau, \rho \in T$ are said to be of the same type if $\mathbf{w}(\tau) = \mathbf{w}(\rho)$ and $u_\tau(\cdot) = u_\rho(\cdot)$. Let $A = \{A_1, A_2, \ldots, A_k, \ldots\}$ be a partition of the set of atoms $T \setminus T_0$ such that $A_k$ contains all the atoms who are of the same type as an atom $\tau_k \in A_k$, for each $k = 1, \ldots, |A|$, where $|A|$ denotes the cardinality of the partition $A$. Moreover, let $T_k$ be the set of the traders $t \in T$ who are of the same type as the atoms in $A_k$, for each $k = 1, \ldots, |A|$. Given a set $T_k$, denote by $\tau_{hk}$ the $h$-th atom belonging to the set $T_k$, for each $h = 1, \ldots, |A_k|$, where $|A_k|$ denotes the cardinality of the set $A_k$. We can now state the two theorems.

**Theorem 2** [Gabszewicz and Mertens (1971)]. *Under Assumptions 1, 2, and 3, if, either* $|A| = 1$ *and* $\sum_{h=1}^{|A_1|} \frac{\mu(\tau_{h1})}{\mu(T_1)} < 1$, *or,* $|A| > 1$ *and* $\sum_{k=1}^{|A|} \sum_{h=1}^{|A_k|} \frac{\mu(\tau_{hk})}{\mu(T_k)} \leq 1$, *then the core coincides with the set of Walras allocations.*

**Theorem 3** [Shitovitz (1973)]. *Under Assumptions 1, 2, and 3, if* $|A| = 1$ *and* $|A_1| \geq 2$, *then the core coincides with the set of Walras allocations.*

Okuno et al. (1980) already showed that the equivalence stated by Theorem 3 (Shitovitz's Theorem B) does not extend to the set of Cournot-Nash allocations, thereby breaking the symmetry of Theorem 1. In the next section, we further investigate the relation between the core and the sets of Walras and Cournot-Nash allocations.

# 4 Some examples, two propositions, and a theorem

In the following example (Example 1 in Shitovitz (1973)), the market of commodity 2 is monopolistic. The example shows that Theorems 2 and 3 cannot be extended to this case as $|A| = 1$, $|A_1| = 1$, and $\frac{\mu(\tau_{11})}{\mu(T_1)} = 1$. Moreover, in this market configuration, the sets of Walras and Cournot-Nash allocations are disjoint as there is no Cournot-Nash equilibrium.

**Example 1.** *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3.* $T_0 = [0, 1]$, $A_1 = \{2\}$, $T_0$ *is taken with Lebesgue measure,* $\mu(2) = 1$, $\mathbf{w}(t) = (4, 0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in T_0$, $\mathbf{w}(2) = (0, 4)$, $u_2(x) = \sqrt{x^1} + \sqrt{x^2}$. *Then, there is an allocation in the core, which is not a Walras allocation, and there is no Cournot-Nash allocation.*

**Proof.** The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = (1,1)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (2,2)$, for each $t \in T_0$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (2,2)$. As shown by Shitovitz (1973), the allocation $\tilde{\mathbf{x}}$ such that $(\tilde{\mathbf{x}}^1(t), \tilde{\mathbf{x}}^2(t)) = (1,1)$, for each $t \in T_0$, $(\tilde{\mathbf{x}}^1(2), \tilde{\mathbf{x}}^2(2)) = (3,3)$ is in the core but it is not a Walras allocation. Suppose that there is a Cournot-Nash allocation $\hat{\mathbf{x}}$. Then, there is a strategy selection $\hat{\mathbf{b}}$ which is a Cournot-Nash equilibrium and which is such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. In particular, $\mathbf{x}(2, \hat{\mathbf{b}}(2), p(\hat{\mathbf{b}})) = (\bar{\hat{\mathbf{b}}}_{12}, 4 - \hat{\mathbf{b}}_{21}(2))$. Let $b'(2)$ be a strategy such that $0 < b'_{21}(2) < \hat{\mathbf{b}}_{21}(2)$. Then,

$$u_2(\mathbf{x}(2, \hat{\mathbf{b}} \setminus b'(2), p(\hat{\mathbf{b}} \setminus b'(2)))) > u_2(\mathbf{x}(2, \hat{\mathbf{b}}(2), p(\hat{\mathbf{b}}))),$$

as $\mathbf{x}(2, \hat{\mathbf{b}} \setminus b'(2), p(\hat{\mathbf{b}} \setminus b'(2))) = (\bar{\hat{\mathbf{b}}}_{12}, 4 - b'_{21}(2))$ and $u_2(\cdot)$ is strongly monotone, a contradiction. Then, there is no Cournot-Nash allocation. ■

In the following example, all traders have the same utility function as in Example 1 but a competitive fringe competes with the monopolist in the market for commodity 2. The core coincides with the set of Walras allocations as the assumptions of Theorem 2 are satisfied but no Cournot-Nash allocation is in the core.

**Example 2.** *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3.* $T_0 = [0,1]$, $A_1 = \{2\}$, $T_0$ *is taken with Lebesgue measure,* $\mu(2) = 1$, $\mathbf{w}(t) = (4,0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0,4)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in [\frac{1}{2}, 1]$, $\mathbf{w}(2) = (0,4)$, $u_2(x) = \sqrt{x^1} + \sqrt{x^2}$. *Then, there is a unique allocation in the core which is also the unique Walras allocation but which is not a Cournot-Nash allocation.*

**Proof.** The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = (\sqrt{3}, 1)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{4}{\sqrt{3}+1}, \frac{12}{\sqrt{3}+1})$, for each $t \in [0, \frac{1}{2}]$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{4}{\sqrt{3}+3}, \frac{4\sqrt{3}}{\sqrt{3}+1})$, for each $t \in [\frac{1}{2}, 1]$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (\frac{4}{\sqrt{3}+3}, \frac{4\sqrt{3}}{\sqrt{3}+1})$. Then, by Theorem 2, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$, $|A_1| = 1$, and $\frac{\mu(\tau_{11})}{\mu(T_1)} < 1$. Suppose that $\mathbf{x}^*$ is also a Cournot-Nash allocation. Then, there is a strategy selection $\mathbf{b}^*$ which is a Cournot-Nash equilibrium and which is such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. But then, $\mathbf{b}^*$ must be such that $\mathbf{b}^*_{12}(t) = \frac{4\sqrt{3}}{\sqrt{3}+1}$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{b}^*_{21}(t) = \frac{4}{\sqrt{3}+1}$, for each $t \in [\frac{1}{2}, 1]$, $\mathbf{b}^*_{21}(2) = \frac{4}{\sqrt{3}+1}$. However, it is straightforward to verify that $\mathbf{b}^*(2) \notin \arg\max\{u_2(\mathbf{x}(t, \mathbf{b}^* \setminus b(2), p(\mathbf{b}^* \setminus b(2)))) :$

$b(2) \in \mathbf{B}(2)\}$, a contradiction. Then, the unique Walras allocation is not a Cournot-Nash allocation. ■

In the following example, all traders have the same utility function as in Example 1 but there are two oligopolists of the same type in the market for commodity 2. The core coincides with the set of Walras allocations as the assumptions of Theorem 3 are satisfied but no Cournot-Nash allocation is in the core.

**Example 3.** *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3.* $T_0 = [0,1]$, $A_1 = \{2,3\}$, $T_0$ *is taken with Lebesgue measure,* $\mu(2) = \mu(3) = 1$, $\mathbf{w}(t) = (4,0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in T_0$, $\mathbf{w}(2) = \mathbf{w}(3) = (0,4)$, $u_2(x) = u_3(x) = \sqrt{x^1} + \sqrt{x^2}$. *Then, there is a unique allocation in the core which is also the unique Walras allocation but which is not a Cournot-Nash allocation.*

**Proof.** The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = (\sqrt{2}, 1)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{4}{\sqrt{2}+1}, \frac{8}{\sqrt{2}+1})$, for each $t \in T_0$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (\mathbf{x}^{*1}(3), \mathbf{x}^{*2}(3)) = (\frac{4}{\sqrt{2}+2}, \frac{4\sqrt{2}}{\sqrt{2}+1})$. Then, by Theorem 3, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$ and $|A_1| = 2$. Suppose that $\mathbf{x}^*$ is also a Cournot-Nash allocation. Then, there is a strategy selection $\mathbf{b}^*$ which is a Cournot-Nash equilibrium and which is such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. But then, $\mathbf{b}^*$ must be such that $\mathbf{b}^*_{12}(t) = \frac{4\sqrt{2}}{\sqrt{2}+1}$, for each $t \in T_0$, $\mathbf{b}^*_{21}(2) = \mathbf{b}^*_{21}(3) = \frac{4}{\sqrt{2}+1}$. However, it is straightforward to verify that $\mathbf{b}^*(2) \notin \arg\max\{u_2(\mathbf{x}(t, \mathbf{b}^* \setminus b(2), p(\mathbf{b}^* \setminus b(2)))) : b(2) \in \mathbf{B}(2)\}$, a contradiction. Then, the unique Walras allocation is not a Cournot-Nash allocation. ■

In Examples 2 and 3, there are atoms who demand a strictly positive amount of both commodities at a Walras equilibrium and the sets of Walras and Cournot-Nash allocations are disjoint. The following proposition generalizes these examples providing a necessary condition for a Walras allocation to be a Cournot-Nash allocation. In order to state the proposition, we need a further assumption on traders' utility functions.

**Assumption 4.** $u_t : R^2_+ \to R$ *is differentiable, for each* $t \in T \setminus T_0$.[5]

[5] In this assumption, differentiability should be implicitly understood to include the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

**Proposition 1.** *Under Assumptions 1, 2, 3, and 4, if the pair* $(p^*, \mathbf{x}^*)$ *is a Walras equilibrium such that* $\mathbf{x}^*(\tau) \gg 0$, *for an atom* $\tau \in T \setminus T_0$, *then* $\mathbf{x}^*$ *is not a Cournot-Nash allocation.*

**Proof**. Suppose that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium such that $\mathbf{x}^*(\tau) \gg 0$, for an atom $\tau \in T \setminus T_0$. Moreover, suppose that $\mathbf{x}^*$ is a Cournot-Nash allocation. Then, there is a strategy selection $\mathbf{b}^*$ such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$, where $\mathbf{b}^*$ is a Cournot-Nash equilibrium. Since, given a trader $t \in T$, $p(\mathbf{b}^*)\mathbf{x}^*(t) = p(\mathbf{b}^*)\mathbf{w}(t)$ and $p^*$ is the unique price vector such that $p^*\mathbf{x}^*(t) = p^*\mathbf{w}(t)$, $p^* = p(\mathbf{b}^*)$. Consider the atom $\tau \in T \setminus T_0$ and assume, without loss of generality, that $\mathbf{w}^1(\tau) = 0$ and $\mathbf{w}^2(\tau) > 0$. At a Cournot-Nash equilibrium, for the atom $\tau$, the marginal rate of substitution must be equal to the marginal rate at which he can trade off commodity 1 for commodity 2 (see Okuno et al. (1980)). Moreover, at a Walras equilibrium, the marginal rate of substitution must be equal to the relative price of commodity 1 in terms of commodity 2. These two conditions are expressed by the following equations:

$$\frac{dx^2}{dx^1} = -\frac{p^{*1}}{p^{*2}} \frac{\overline{\mathbf{b}^*}_{21}}{\overline{\mathbf{b}^*}_{21} - \mathbf{b}^*_{21}(\tau)\mu(\tau)} = -\frac{p^{*1}}{p^{*2}}.$$

Then, we must have $\mathbf{b}^*_{21}(\tau) = 0$. But then, $(\mathbf{x}^{*1}(\tau), \mathbf{x}^{*2}(\tau)) = (0, \mathbf{w}^2(\tau))$, a contradiction. Hence, $\mathbf{x}^*$ is not a Cournot-Nash allocation. ∎

The following example differs from Example 2 only in that the monopolist and the competitive fringe have quasi-linear utility functions. It shows that, under the assumptions of Theorem 2, the converse of Proposition 1 does not hold. At the unique Walras equilibrium, both the monopolist and the competitive fringe demand a null amount of commodity 2 and this unique Walras allocation is also the unique allocation in the core but it is not a Cournot-Nash allocation.

**Example 4.** *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3.* $T_0 = [0, 1]$, $A_1 = \{2\}$, $T_0$ *is taken with Lebesgue measure,* $\mu(2) = 1$, $\mathbf{w}(t) = (4, 0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0, 4)$, $u_t(x) = \sqrt{x^1} + \frac{1}{10}x^2$, *for each* $t \in [\frac{1}{2}, 1]$, $\mathbf{w}(2) = (0, 4)$, $u_2(x) = \sqrt{x^1} + \frac{1}{10}x^2$. *Then, there is a unique allocation in the core which is also the unique Walras allocation but which is not a Cournot-Nash allocation.*

**Proof.** The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = (\frac{\sqrt{21}+3}{2}, 1)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{8}{\sqrt{21}+5}, 12)$, for each $t \in [0, \frac{1}{2}]$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) =$

$(\frac{8}{\sqrt{21}+3}, 0)$, for each $t \in [\frac{1}{2}, 1]$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (\frac{8}{\sqrt{21}+3}, 0)$. Then, by Theorem 2, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$, $|A_1| = 1$, and $\frac{\mu(\tau_{11})}{\mu(T_1)} < 1$. Suppose that $\mathbf{x}^*$ is also a Cournot-Nash allocation. Then, there is a strategy selection $\mathbf{b}^*$ which is a Cournot-Nash equilibrium and which is such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. But then, $\mathbf{b}^*$ must be such that $\mathbf{b}^*_{12}(t) = \frac{4\sqrt{21}+12}{\sqrt{21}+5}$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{b}^*_{21}(t) = 4$, for each $t \in [\frac{1}{2}, 1]$, $\mathbf{b}^*_{12}(2) = 4$. However, it is straightforward to verify that $\mathbf{b}^*(2) \notin \arg\max\{u_2(\mathbf{x}(t, \mathbf{b}^* \setminus b(2), p(\mathbf{b}^* \setminus b(2)))) : b(2) \in \mathbf{B}(2)\}$, a contradiction. Then, the unique Walras allocation is not a Cournot-Nash allocation. ■

The following example differs from Example 3 only in that the two oligopolists have quasi-linear utility functions. It shows that, under the assumptions of Theorem 3, the converse of Proposition 1 does not hold. At the unique Walras equilibrium, the two oligopolists demand a null amount of commodity 2 and this unique Walras allocation is also the unique allocation in the core but it is not a Cournot-Nash allocation.

**Example 5.** *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3.* $T_0 = [0, 1]$, $A_1 = \{2, 3\}$, $T_0$ *is taken with Lebesgue measure,* $\mu(2) = \mu(3) = 1$, $\mathbf{w}(t) = (4, 0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in T_0$, $\mathbf{w}(2) = \mathbf{w}(3) = (0, 4)$, $u_2(x) = u_3(x) = \sqrt{x^1} + \frac{1}{10}x^2$. *Then, there is a unique allocation in the core which is also the unique Walras allocation but which is not a Cournot-Nash allocation.*

**Proof.** The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = (\sqrt{3} + 1, 1)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{4}{\sqrt{3}+2}, 8)$, for each $t \in T_0$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (\mathbf{x}^{*1}(3), \mathbf{x}^{*2}(3)) = (\frac{4}{\sqrt{3}+1}, 0)$. Then, by Theorem 3, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$ and $|A_1| = 2$. Suppose that $\mathbf{x}^*$ is also a Cournot-Nash allocation. Then, there is a strategy selection $\mathbf{b}^*$ which is a Cournot-Nash equilibrium and which is such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. But then, $\mathbf{b}^*$ must be such that $\mathbf{b}^*_{12}(t) = \frac{4\sqrt{3}+4}{\sqrt{3}+2}$, for each $t \in T_0$, $\mathbf{b}^*_{21}(2) = \mathbf{b}^*_{21}(3) = 4$. However, it is straightforward to verify that $\mathbf{b}^*(2) \notin \arg\max\{u_2(\mathbf{x}(t, \mathbf{b}^* \setminus b(2), p(\mathbf{b}^* \setminus b(2)))) : b(2) \in \mathbf{B}(2)\}$, a contradiction. Then, the unique Walras allocation is not a Cournot-Nash allocation. ■

We now address the question whether, in mixed exchange economies, an equivalence, or at least a nonempty intersection, between the sets of Walras and Cournot-Nash allocations may hold. The following example differs from

Example 4 only for the lower "weight" of commodity 2 for traders who have quasi-linear utility functions. At the unique Walras equilibrium, both the monopolist and the competitive fringe demand a null amount of commodity 2 and this unique Walras allocation is also the unique allocation in the core and the unique Cournot-Nash allocation.

**Example 6.** *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3.* $T_0 = [0, 1]$, $A_1 = \{2\}$, $T_0$ *is taken with Lebesgue measure,* $\mu(2) = 1$, $\mathbf{w}(t) = (4, 0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0, 4)$, $u_t(x) = \sqrt{x^1} + \frac{1}{30}x^2$, *for each* $t \in [\frac{1}{2}, 1]$, $\mathbf{w}(2) = (0, 4)$, $u_2(x) = \sqrt{x^1} + \frac{1}{30}x^2$. *Then, there is a unique allocation in the core which is also the unique Walras allocation and the unique Cournot-Nash allocation.*

**Proof.** The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = (\frac{\sqrt{21}+3}{2}, 1)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{8}{\sqrt{21}+5}, 12)$, for each $t \in [0, \frac{1}{2}]$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{8}{\sqrt{21}+3}, 0)$, for each $t \in [\frac{1}{2}, 1]$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (\frac{8}{\sqrt{21}+3}, 0)$. Then, by Theorem 2, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$, $|A_1| = 1$, and $\frac{\mu(\tau_{11})}{\mu(T_1)} < 1$. Moreover, the strategy selection $\mathbf{b}^*$, where $\mathbf{b}^*_{12}(t) = \frac{4\sqrt{21}+12}{\sqrt{21}+5}$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{b}^*_{21}(t) = 4$, for each $t \in [\frac{1}{2}, 1]$, $\mathbf{b}^*_{21}(2) = 4$, is the unique Cournot-Nash equilibrium and $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. Then, the unique Walras allocation is also the unique Cournot-Nash allocation. ∎

The following example differs from Example 5 only for the lower "weight" of commodity 2 for traders who have quasi-linear utility functions. At the unique Walras equilibrium, the two oligopolists demand a null amount of commodity 2 and this unique Walras allocation is also the unique allocation in the core and the unique Cournot-Nash allocation.

**Example 7.** *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3.* $T_0 = [0, 1]$, $A_1 = \{2, 3\}$, $T_0$ *is taken with Lebesgue measure,* $\mu(2) = \mu(3) = 1$, $\mathbf{w}(t) = (4, 0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in T_0$, $\mathbf{w}(2) = \mathbf{w}(3) = (0, 4)$, $u_2(x) = u_3(x) = \sqrt{x^1} + \frac{1}{30}x^2$. *Then, there is a unique allocation in the core which is also the unique Walras allocation and the unique Cournot-Nash allocation.*

**Proof.** The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = (\sqrt{3} + 1, 1)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{4}{\sqrt{3}+2}, 8)$, for each $t \in T_0$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (\mathbf{x}^{*1}(3), \mathbf{x}^{*2}(3)) = (\frac{4}{\sqrt{3}+1}, 0)$. Then, by Theorem 3, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$ and $|A_1| = 2$.

Moreover, the strategy selection $\mathbf{b}^*$, where $\mathbf{b}^*_{12}(t) = \frac{4\sqrt{3}+4}{\sqrt{3}+2}$, for each $t \in T_0$, $\mathbf{b}^*_{21}(2) = \mathbf{b}^*_{21}(3) = 4$, is the unique Cournot-Nash equilibrium and $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. Then, the unique Walras allocation is also the unique Cournot-Nash allocation. ∎

Examples 6 and 7 differ from Examples 4 and 5 as, in the latter, all atoms who hold commodity 2 demand a null amount of this commodity at a Walras equilibrium but not at a Cournot-Nash equilibrium whereas, in the former, they also demand a null amount of commodity 2 at a Cournot-Nash equilibrium. The following theorem generalizes Examples 6 and 7 as it shows that demanding a null amount of one of the two commodities by all the atoms is a necessary and sufficient condition for a Cournot-Nash allocation to be a Walras allocation.

**Theorem 4.** *Under Assumptions 1, 2, 3, and 4, let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium if and only if $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$, for each $t \in T \setminus T_0$.*

**Proof**. Let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Suppose that the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. Moreover, suppose that $\hat{\mathbf{x}}(\tau) \gg 0$, for an atom $\tau \in T \setminus T_0$. Then, $\hat{\mathbf{x}}$ is not a Cournot-Nash allocation, by Proposition 1, a contradiction. Hence, $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$, for each $t \in T \setminus T_0$. Conversely, suppose that $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$, for each $t \in T \setminus T_0$. Consider an atom $\tau \in T \setminus T_0$ and assume, without loss of generality, that $\mathbf{w}^1(\tau) = 0$ and $\mathbf{w}^2(\tau) > 0$. Consider the case where $\hat{\mathbf{x}}^1(\tau) = 0$. Then, $\hat{\mathbf{b}}_{21}(\tau) = 0$ and $\hat{\mathbf{x}}(\tau) = (0, \mathbf{w}^2(\tau))$. We have that $\hat{p}\hat{\mathbf{x}}(\tau) = \hat{p}\mathbf{w}(\tau)$ since

$$\hat{p}^1\hat{\mathbf{x}}^1(\tau) + \hat{p}^2\hat{\mathbf{x}}^2(\tau) = \hat{p}^1 0 + \hat{p}^2(\mathbf{w}^2(\tau) - 0) = \hat{p}^2\mathbf{w}^2(\tau).$$

Let $\hat{x}^2(x^1)$ be a function such that $u_\tau(x^1, \hat{x}^2(x^1)) \equiv u_\tau(\hat{\mathbf{x}}(\tau))$, for each $0 \leq x^1 \leq \mathbf{w}^2(\tau)\frac{\hat{p}^2}{\hat{p}^1}$. We have that

$$\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^1} \frac{\bar{\hat{\mathbf{b}}}_{21} - \hat{\mathbf{b}}_{21}(\tau)\mu(\tau)}{\bar{\hat{\mathbf{b}}}_{21}} \frac{\hat{p}^2}{\hat{p}^1} - \frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} \leq 0$$

as $\hat{\mathbf{b}}_{21}(\tau) = 0$. Then,

$$\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^1} \frac{\hat{p}^2}{\hat{p}^1} - \frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} \leq 0$$

as $\frac{\bar{\hat{\mathbf{b}}}_{21}-0}{\bar{\hat{\mathbf{b}}}_{21}} = 1$. But then, $\frac{d\hat{x}^2(0)}{dx^1} \geq -\frac{\hat{p}^1}{\hat{p}^2}$. Consider the case where $\frac{d\hat{x}^2(0)}{dx^1} = -\frac{\hat{p}^1}{\hat{p}^2}$. Then, $u_\tau(\hat{\mathbf{x}}(\tau)) \geq u_\tau(y)$ for all $y \in \{x \in R^2_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$, as $u_\tau(\cdot)$ is quasi-concave, by Assumption 2. Consider now the case where $\frac{d\hat{x}^2(0)}{dx^1} > -\frac{\hat{p}^1}{\hat{p}^2}$. Then, $\frac{d\hat{x}^2(x^1)}{dx^1} > -\frac{\hat{p}^1}{\hat{p}^2}$, for each $0 \leq x^1 \leq \mathbf{w}^2(\tau)\frac{\hat{p}^2}{\hat{p}^1}$, as $u_\tau(\cdot)$ is quasi-concave, by Assumption 2. Suppose that there exists a commodity bundle $\tilde{x} \in \{x \in R^2_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$ such that $u_\tau(\tilde{x}) > u_\tau(\hat{\mathbf{x}}(\tau))$. Then, $\tilde{x}^2 > \hat{x}^2(\tilde{x}^1)$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2. But then, by the Mean Value Theorem, there exists some $\bar{x}^1$ such that $0 < \bar{x}^1 < \tilde{x}^1$ and such that

$$\frac{d\hat{x}^2(\bar{x}^1)}{dx^1} = \frac{\hat{x}^2(0) - \hat{x}^2(\tilde{x}^1)}{0 - \tilde{x}^1} < -\frac{\hat{p}^1}{\hat{p}^2},$$

a contradiction. Therefore, $u_\tau(\hat{\mathbf{x}}(\tau)) \geq u_\tau(y)$ for all $y \in \{x \in R^2_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$. Consider now the case where $\hat{\mathbf{x}}^2(\tau) = 0$. Then, $\hat{\mathbf{b}}_{21}(\tau) = \mathbf{w}^2(\tau)$ and $\hat{\mathbf{x}}(\tau) = (\mathbf{w}^2(\tau)\frac{\hat{p}^2}{\hat{p}^1}, 0)$. We have that $\hat{p}\hat{\mathbf{x}}(\tau) = \hat{p}\mathbf{w}(\tau)$ since

$$\hat{p}^1\hat{\mathbf{x}}^1(\tau) + \hat{p}^2\hat{\mathbf{x}}^2(\tau) = \hat{p}^1\mathbf{w}^2(\tau)\frac{\hat{p}^2}{\hat{p}^1} + \hat{p}^2(\mathbf{w}^2(\tau) - \mathbf{w}^2(\tau)) = \hat{p}^2\mathbf{w}^2(\tau).$$

Let $\hat{x}^2(x^1)$ be a function such that $u_\tau(x^1, \hat{x}^2(x^1)) \equiv u_\tau(\hat{\mathbf{x}}(\tau))$, for each $0 \leq x^1 \leq \mathbf{w}^2(\tau)\frac{\hat{p}^2}{\hat{p}^1}$. We have that

$$\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^1}\frac{\bar{\hat{\mathbf{b}}}_{21} - \hat{\mathbf{b}}_{21}(\tau)\mu(\tau)}{\bar{\hat{\mathbf{b}}}_{21}}\frac{\hat{p}^2}{\hat{p}^1} - \frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} \geq 0$$

as $\hat{\mathbf{b}}_{21}(\tau) = \mathbf{w}^2(\tau)$. Then,

$$\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^1}\frac{\hat{p}^2}{\hat{p}^1} - \frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} > 0$$

as $\frac{\bar{\hat{\mathbf{b}}}_{21}-\mathbf{w}^2(\tau)\mu(\tau)}{\bar{\hat{\mathbf{b}}}_{21}} < 1$. But then, $\frac{d\hat{x}^2(x^1)}{dx^1} < -\frac{\hat{p}^1}{\hat{p}^2}$, for each $0 \leq x^1 \leq \mathbf{w}^2(\tau)\frac{\hat{p}^2}{\hat{p}^1}$, as $u_\tau(\cdot)$ is quasi-concave, by Assumption 2. Suppose that there exists a commodity bundle $\tilde{x} \in \{x \in R^2_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$ such that $u_\tau(\tilde{x}) > u_\tau(\hat{\mathbf{x}}(\tau))$. Then, $\tilde{x}^2 > \hat{x}^2(\tilde{x}^1)$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2. But then, by the Mean Value Theorem, there exists some $\bar{x}^1$ such that $\tilde{x}^1 < \bar{x}^1 < \mathbf{w}^2(\tau)\frac{\hat{p}^2}{\hat{p}^1}$ and such that

$$\frac{d\hat{x}^2(\bar{x}^1)}{dx^1} = \frac{\hat{x}^2(\tilde{x}^1) - \hat{x}^2(\mathbf{w}^2(\tau)\frac{\hat{p}^2}{\hat{p}^1})}{\tilde{x}^1 - \mathbf{w}^2(\tau)\frac{\hat{p}^2}{\hat{p}^1}} > -\frac{\hat{p}^1}{\hat{p}^2},$$

a contradiction. Therefore, $u_\tau(\hat{\mathbf{x}}(\tau)) \geq u_\tau(y)$ for all $y \in \{x \in R^2_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$. We then conclude that $\hat{p}\hat{\mathbf{x}}(t) = \hat{p}\mathbf{w}(t)$ and $u_t(\hat{\mathbf{x}}(t)) \geq u_t(y)$ for all $y \in \{x \in R^2_+ : \hat{p}x = \hat{p}\mathbf{w}(t)\}$, for each $t \in T \setminus T_0$. Moreover, it is straightforward to show (see, for instance, Proposition 3 in Busetto et al. (2013)) that $\hat{p}\hat{\mathbf{x}}(t) = \hat{p}\mathbf{w}(t)$ and $u_t(\hat{\mathbf{x}}(t)) \geq u_t(y)$ for all $y \in \{x \in R^2_+ : \hat{p}x = \hat{p}\mathbf{w}(t)\}$, for each $t \in T_0$. Hence, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. ■

Examples 6 and 7 show that Theorem 4 is non-vacuous when atoms demand, at a Cournot-Nash equilibrium, a null amount of the commodity they hold. The following two examples show that it is also non-vacuous when atoms demand, at a Cournot-Nash equilibrium, a null amount of the commodity they do not hold.

The structure of the following example differs from that of Example 6 for a further competitive fringe which holds commodity 2 and is not of the same type as the monopolist. At the unique Walras equilibrium, both the monopolist and the competitive fringe with traders of the same type as the monopolist demand a null amount of commodity 1 and this unique Walras allocation is also the unique allocation in the core and the unique Cournot-Nash allocation.

**Example 8.** *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3.* $T_0 = [0, 1]$, $A_1 = \{2\}$, *$T_0$ is taken with Lebesgue measure,* $\mu(2) = 1$, $\mathbf{w}(t) = (4, 0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in [0, \frac{1}{3}]$, $\mathbf{w}(t) = (0, 4)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in [\frac{1}{3}, \frac{2}{3}]$, $\mathbf{w}(t) = (0, 4)$, $u_t(x) = \frac{1}{4}x^1 + \sqrt{x^2}$, *for each* $t \in [\frac{2}{3}, 1]$, $\mathbf{w}(2) = (0, 4)$, $u_2(x) = \frac{1}{4}x^1 + \sqrt{x^2}$. *Then, there is a unique allocation in the core which is also the unique Walras allocation and the unique Cournot-Nash allocation.*

**Proof.** The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = (1, 1)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (2, 2)$, for each $t \in [0, \frac{1}{3}]$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (2, 2)$, for each $t \in [\frac{1}{3}, \frac{2}{3}]$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (0, 4)$, for each $t \in [\frac{2}{3}, 1]$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (0, 4)$. Then, by Theorem 2, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$, $|A_1| = 1$, and $\frac{\mu(\tau_{11})}{\mu(T_1)} < 1$. Moreover, the strategy selection $\mathbf{b}^*$, where $\mathbf{b}^*_{12}(t) = 2$, for each $t \in [0, \frac{1}{3}]$, $\mathbf{b}^*_{21}(t) = 2$, for each $t \in [\frac{1}{3}, \frac{2}{3}]$, $\mathbf{b}^*_{21}(t) = 0$, for each $t \in [\frac{2}{3}, 1]$, $\mathbf{b}^*_{21}(2) = 0$, is the unique Cournot-Nash equilibrium and $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. Then, the unique Walras allocation is also the unique Cournot-Nash allocation. ■

The structure of the following example differs from that of Example 7 for a further competitive fringe which holds commodity 2 and is not of the same type as the two oligopolists. At the unique Walras equilibrium, the two oligopolists demand a null amount of commodity 1 and this unique Walras allocation is also the unique allocation in the core and the unique Cournot-Nash allocation.

**Example 9.** *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3.* $T_0 = [0,1]$, $A_1 = \{2,3\}$, $T_0$ *is taken with Lebesgue measure,* $\mu(2) = \mu(3) = 1$, $\mathbf{w}(t) = (4,0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0,4)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in [\frac{1}{2}, 1]$, $\mathbf{w}(2) = \mathbf{w}(3) = (0,4)$, $u_2(x) = u_3(x) = \frac{1}{4}x^1 + \sqrt{x^2}$. *Then, there is a unique allocation in the core which is also the unique Walras allocation and the unique Cournot-Nash allocation.*

**Proof.** The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = (1,1)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (2,2)$, for each $t \in [0, \frac{1}{2}]$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (2,2)$, for each $t \in [\frac{1}{2}, 1]$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (\mathbf{x}^{*1}(3), \mathbf{x}^{*2}(3)) = (0,4)$. Then, by Theorem 3, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$ and $|A_1| = 2$. Moreover, the strategy selection $\mathbf{b}^*$, where $\mathbf{b}^*_{12}(t) = 2$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{b}^*_{21}(t) = 2$, for each $t \in [\frac{1}{2}, 1]$, $\mathbf{b}^*_{21}(2) = \mathbf{b}^*_{21}(3) = 0$, is the unique Cournot-Nash equilibrium and $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. Then, the unique Walras allocation is also the unique Cournot-Nash allocation. ■

In all the examples of this section, preferences are represented by additively separable utility functions, i.e., utility functions of the form $u(x) = v^1(x^1) + v^2(x^2)$, for each $x \in R^2_+$. We conclude this section by providing a necessary condition for Theorem 4 to hold when atoms' preferences are represented by an additively separable utility function.

**Proposition 2.** *Under Assumptions 1, 2, 3, and 4, let* $\hat{\mathbf{b}}$ *be a Cournot-Nash equilibrium and let* $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, *for each* $t \in T$. *Then, for each* $t \in T \setminus T_0$ *such that* $u_t(x) = v^1_t(x^1) + v^2_t(x^2)$, $\hat{\mathbf{x}}^1(t) = 0$ *only if* $-\frac{\partial u_t(0,x^2)}{\partial x^1} / \frac{\partial u_t(0,x^2)}{\partial x^2} > -\infty$, *for each* $x^2 \in R_+$, *and* $\hat{\mathbf{x}}^2(t) = 0$ *only if* $-\frac{\partial u_t(x^1,0)}{\partial x^1} / \frac{\partial u_t(x^1,0)}{\partial x^2} < 0$, *for each* $x^1 \in R_+$.

**Proof**. Let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Moreover, let $\hat{p} = p(\hat{\mathbf{b}})$. Consider an atom $\tau \in T \setminus T_0$ such that $u_\tau(x) = v^1_\tau(x^1) + v^2_\tau(x^2)$. Suppose that $\hat{\mathbf{x}}^1(\tau) = 0$. By the same argument used in the proof of Theorem 4, it follows that

$$-\frac{\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^1}}{\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2}} \geq -\frac{\hat{p}^1}{\hat{p}^2}.$$

Then,

$$-\frac{\frac{\partial u_\tau(0,\hat{\mathbf{x}}^2(\tau))}{\partial x^1}}{\frac{\partial u_\tau(0,\hat{\mathbf{x}}^2(\tau))}{\partial x^2}} > -\infty.$$

But then, $\frac{\partial u_\tau(0,\hat{\mathbf{x}}^2(\tau))}{\partial x^1} = \frac{\partial v^1_\tau(0)}{\partial x^1} = \frac{\partial u_\tau(0,x^2)}{\partial x^1} < +\infty$, for each $x^2 \in R_+$. Moreover, $\frac{\partial u_\tau(0,x^2)}{\partial x^2} > 0$, for each $x^2 \in R_+$, as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2. Therefore, $-\frac{\partial u_\tau(0,x^2)}{\partial x^1} / \frac{\partial u_\tau(0,x^2)}{\partial x^2} > -\infty$, for each $x^2 \in R_+$. Suppose that $\hat{\mathbf{x}}^2(\tau) = 0$. By the same argument used in the proof of Theorem 4, it follows that

$$-\frac{\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^1}}{\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2}} < -\frac{\hat{p}^1}{\hat{p}^2}.$$

Then,

$$-\frac{\frac{\partial u_\tau(\hat{\mathbf{x}}^1(\tau),0)}{\partial x^1}}{\frac{\partial u_\tau(\hat{\mathbf{x}}^1(\tau),0)}{\partial x^2}} < 0.$$

But then, $\frac{\partial u_\tau(\hat{\mathbf{x}}^1(\tau),0)}{\partial x^2} = \frac{\partial v^2_\tau(0)}{\partial x^2} = \frac{\partial u_\tau(x^1,0)}{\partial x^2} < +\infty$, for each $x^1 \in R_+$. Moreover, $\frac{\partial u_\tau(x^1,0)}{\partial x^1} > 0$, for each $x^1 \in R_+$, as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2. Therefore, $-\frac{\partial u_\tau(x^1,0)}{\partial x^1} / \frac{\partial u_\tau(x^1,0)}{\partial x^2} < 0$, for each $x^1 \in R_+$. Hence, for each $t \in T \setminus T_0$ such that $u_t(x) = v^1_t(x^1) + v^2_t(x^2)$, $\hat{\mathbf{x}}^1(t) = 0$ only if $-\frac{\partial u_t(0,x^2)}{\partial x^1} / \frac{\partial u_t(0,x^2)}{\partial x^2} > -\infty$, for each $x^2 \in R_+$, and $\hat{\mathbf{x}}^2(t) = 0$ only if $-\frac{\partial u_t(x^1,0)}{\partial x^1} / \frac{\partial u_t(x^1,0)}{\partial x^2} < 0$, for each $x^1 \in R_+$. ■

# 5 Conclusion

In this paper, we have reconsidered, in the framework of bilateral oligopoly, the problem raised by Okuno et al. (1980) about the noncooperative foundation of oligopolistic behavior in general equilibrium. We can now summarize the implications of the previous analysis. The condition which requires that the atoms are not "too" big, introduced by Gabszewicz and Mertens (1971), is not necessary for the equivalence between the core and the set of Walras allocations, as shown by Theorem 3, but it is sufficient for this equivalence, by Theorem 2; moreover, it is neither necessary nor sufficient for a nonempty intersection between the sets of Walras and Cournot-Nash allocations as shown, respectively, by Examples 7 and 4. The condition which requires that there are only atoms of the same type, introduced by Shitovitz

(1973), is not necessary for the equivalence between the core and the set of Walras allocations, as shown by Theorem 2, but it is sufficient for this equivalence, by Theorem 3; moreover, it is neither necessary nor sufficient for a nonempty intersection between the sets of Walras and Cournot-Nash allocations as shown, respectively, by Examples 6 and 5. Theorem 4 states that the condition which characterizes the nonempty intersection of the sets of Walras and Cournot-Nash allocations requires that each atom demands a null amount of one commodity. Moreover, Examples 6, 7, 8, and 9 show that this characterization condition is non-vacuous. Proposition 2 provides a rationale for these examples by exhibiting a necessary condition, expressed in terms of bounds on atoms' marginal rates of substitution, for Theorem 4 to hold when atoms' preferences are represented by additively separable utility functions, the same class considered in the examples. We leave as an open problem for further research the generalization of this proposition, namely, the determination of more general assumptions on traders' size, endowments, and preferences under which our characterization condition holds. This analysis could help to understand more deeply which are the differences between atoms' Walrasian behavior in a cooperative and in a noncooperative framework. Some further research should also be devoted to the possibility of generalizing the results achieved in this paper to an exchange economy with more than two commodities.

# Online Appendix

## Discussion of the model

The Shapley window model (Model 1 hereafter) is one of the two models in the literature referring to Shapley and Shubik in which markets are complete, i.e., all commodities can be used for trade. In the other model analyzed by Amir et al. (1990) (Model 2 hereafter), separate exchange and pricing is set up for each pair of commodities, the price in a market being the ratio of the total amount of bids in each of the two commodities which are exchanged in that market. Therefore, in this model, there is one market and one price for each pair of commodities.

In general, with more than two commodities, the sets of Cournot-Nash allocations of the two models differ as, in Model 1, prices are determined for each commodity whereas, in Model 2, prices are determined for each market where pairs of commodities are exchanged and, consequently, they are not necessarily consistent through pairs of markets in which a same commodity is exchanged.

Codognato (2000) started to investigate whether there might be exchange economies in which the sets of Cournot-Nash allocations of the two models coincide. In particular, he proved that the two sets coincide, in exchange economies with an atomless continuum of traders, when prices in Model 2 are consistent at Cournot-Nash equilibria.

Here, we address the question whether this equivalence also holds in the mixed bilateral oligopoly framework, thereby extending to Model 2 the results obtained for Model 1 in Section 4.

We now introduce Model 2.

**Definition 4.** *Given a strategy selection* $\mathbf{b}$, *the* $2 \times 2$ *matrix* $P$ *is said to be the price matrix generated by* $\mathbf{b}$ *if*

$$p_{ij} = \begin{cases} \frac{\bar{\mathbf{b}}_{ij}}{\bar{\mathbf{b}}_{ji}} & \textit{if} \quad \bar{\mathbf{b}}_{ji} \neq 0, \\ 0 & \textit{if} \quad \bar{\mathbf{b}}_{ji} = 0, \end{cases}$$

$i, j = 1, 2$.

We denote by $P(\mathbf{b})$ a function which associates with each strategy selection $\mathbf{b}$ the price matrix $P$ generated by $\mathbf{b}$.

Given a strategy selection $\mathbf{b}$ and a price matrix $P$, consider the assignment determined as follows:

$$\mathbf{x}^j(t, \mathbf{b}(t), P) = \mathbf{w}^j(t) - \sum_{i=1}^{2} \mathbf{b}_{ji}(t) + \sum_{i=1}^{2} \mathbf{b}_{ij}(t) p_{ij},$$

$j = 1, 2$, for each $t \in T$.

Given a strategy selection $\mathbf{b}$ and the function $P(\mathbf{b})$, the traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), P(\mathbf{b})),$$

for each $t \in T$.[6] It is straightforward to show that this assignment is an allocation.

We are now able to define a notion of Cournot-Nash equilibrium for Model 2.

[6]In order to save in notation, with some abuse, we denote by $\mathbf{x}$ both the function $\mathbf{x}(t)$ and the function $\mathbf{x}(t, \mathbf{b}(t), P(\mathbf{b}))$.

**Definition 5.** *A strategy selection $\tilde{\mathbf{b}}$ such that $\overline{\tilde{\mathbf{B}}}$ is irreducible is a Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \tilde{\mathbf{b}}(t), P(\tilde{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, \tilde{\mathbf{b}} \setminus b(t), P(\tilde{\mathbf{b}} \setminus b(t)))),$$

*for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.*[7]

A Cournot-Nash allocation of Model 2 is an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), P(\tilde{\mathbf{b}}))$, for each $t \in T$, where $\tilde{\mathbf{b}}$ is a Cournot-Nash equilibrium of Model 2.

The following lemma establishes a relation between prices and hence traders' final holdings of the two models for strategy selections whose aggregate matrices are irreducible.

**Lemma.** *If $\mathbf{b}$ is a strategy selection such that $\overline{\mathbf{B}}$ is irreducible, then $\frac{p^i(\mathbf{b})}{p^j(\mathbf{b})} = p_{ji}(\mathbf{b})$, $i, j = 1, 2$, and $\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})) = \mathbf{x}(t, \mathbf{b}(t), P(\mathbf{b}))$, for each $t \in T$.*

**Proof**. Let $\mathbf{b}$ be a strategy selection such that $\overline{\mathbf{B}}$ is irreducible. Then, from Definitions 2 and 4, we have $\frac{p^i(\mathbf{b})}{p^j(\mathbf{b})} = \frac{\overline{\mathbf{b}}_{ji}}{\overline{\mathbf{b}}_{ij}} = p_{ji}(\mathbf{b})$, $i, j = 1, 2$. But then, $\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})) = \mathbf{x}(t, \mathbf{b}(t), P(\mathbf{b}))$, for each $t \in T$. ■

The following theorem shows an equivalence between the sets of Cournot-Nash allocations of Model 1 and Model 2.

**Theorem 5.** *Under Assumptions 1, 2, and 3, the sets of Cournot-Nash allocations of Model 1 and Model 2 coincide.*

**Proof**. Let $\hat{\mathbf{x}}$ be a Cournot-Nash allocation of Model 1. Then, there is a strategy selection $\hat{\mathbf{b}}$ which is a Cournot-Nash equilibrium of Model 1 and is such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Suppose that $\hat{\mathbf{x}}$ is not a Cournot-Nash allocation of Model 2. Then, there exists a trader $\tau \in T$ and a strategy $b(\tau) \in \mathbf{B}(\tau)$ such that

$$u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), P(\hat{\mathbf{b}} \setminus b(\tau)))) > u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), P(\hat{\mathbf{b}}))).$$

$\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})) = \mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), P(\mathbf{b}))$, by the Lemma, as $\overline{\hat{\mathbf{B}}}$ is irreducible. Suppose that the matrix $\overline{\hat{\mathbf{B}}} \setminus b(\tau)$ is irreducible. Then, $\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), p(\hat{\mathbf{b}} \setminus b(\tau))) = \mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), P(\hat{\mathbf{b}} \setminus b(\tau)))$, by the Lemma. But then,

$$u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), p(\hat{\mathbf{b}} \setminus b(\tau)))) > u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}}))),$$

[7] According to Amir et al. (1990), the market for commodities 1 and 2 is active if $\overline{\mathbf{b}}_{12} > 0$ and $\overline{\mathbf{b}}_{21} > 0$ and then if and only if $\overline{\tilde{\mathbf{B}}}$ is irreducible. Therefore, as this definition of a Cournot-Nash equilibrium explicitly refers to irreducible matrices, it applies only to Cournot-Nash equilibria at which the market for commodities 1 and 2 is active.

a contradiction. Suppose that the matrix $\bar{\hat{\mathbf{B}}} \setminus b(\tau)$ is not irreducible. Then, $\tau \in T \setminus T_0$ as $\bar{\hat{\mathbf{B}}} \setminus b(t) = \bar{\hat{\mathbf{B}}}$, for each $t \in T_0$. Assume, without loss of generality, that $\mathbf{w}^1(\tau) = 0$ and $\mathbf{w}^2(\tau) > 0$. Then, $\hat{\mathbf{b}}_{21}(\tau) = \bar{\hat{\mathbf{b}}}_{21}$ as the matrix $\bar{\hat{\mathbf{B}}} \setminus b(\tau)$ is not irreducible. But then, $\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})) = (\bar{\hat{\mathbf{b}}}_{12}, \mathbf{w}^2(\tau) - \hat{\mathbf{b}}_{21}(\tau))$. Let $b'(\tau)$ be a strategy such that $0 < b'_{21}(\tau) < \hat{\mathbf{b}}_{21}(\tau)$. Then,

$$u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b'(\tau), p(\hat{\mathbf{b}} \setminus b'(\tau)))) > u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}}))),$$

as $\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b'(\tau), p(\hat{\mathbf{b}} \setminus b'(\tau))) = (\bar{\hat{\mathbf{b}}}_{12}, \mathbf{w}^2(\tau) - b'_{21}(\tau))$ and $u_\tau(\cdot)$ is strongly monotone, by Assumption 2, a contradiction. Therefore, $\hat{\mathbf{x}}$ is a Cournot-Nash allocation of Model 2. Let $\tilde{\mathbf{x}}$ be a Cournot-Nash allocation of Model 2. Suppose that $\tilde{\mathbf{x}}$ is not a Cournot-Nash allocation of Model 1. Then, the previous argument leads, *mutatis mutandis*, to the same kind of contradictions. Therefore, $\tilde{\mathbf{x}}$ is a Cournot-Nash allocation of Model 1. Hence, the sets of Cournot-Nash allocations of Model 1 and Model 2 coincide. ■

The following corollary shows that Theorem 4 extends, *mutatis mutandis*, to Model 2.

**Corollary.** *Under Assumptions 1, 2, 3, and 4, let* $\tilde{\mathbf{b}}$ *be a Cournot-Nash equilibrium of Model 2 and let* $\tilde{p} = (\bar{\tilde{\mathbf{b}}}_{21}, \bar{\tilde{\mathbf{b}}}_{12})$ *and* $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), p(\tilde{\mathbf{b}}))$, *for each* $t \in T$. *Then, the pair* $(\tilde{p}, \tilde{\mathbf{x}})$ *is a Walras equilibrium if and only if* $\tilde{\mathbf{x}}^1(t) = 0$ *or* $\tilde{\mathbf{x}}^2(t) = 0$, *for each* $t \in T \setminus T_0$.

**Proof**. Let $\tilde{\mathbf{b}}$ be a Cournot-Nash equilibrium of Model 2 and let $\tilde{p} = (\bar{\tilde{\mathbf{b}}}_{21}, \bar{\tilde{\mathbf{b}}}_{12})$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), p(\tilde{\mathbf{b}}))$, for each $t \in T$. $\tilde{\mathbf{b}}$ is a Cournot-Nash equilibrium of Model 1, by Theorem 5, and $\tilde{p} = (\bar{\tilde{\mathbf{b}}}_{21}, \bar{\tilde{\mathbf{b}}}_{12}) = (p^1(\tilde{\mathbf{b}}), p^2(\tilde{\mathbf{b}}))$, by Definition 2. Hence, by Theorem 4, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium if and only if $\tilde{\mathbf{x}}^1(t) = 0$ or $\tilde{\mathbf{x}}^2(t) = 0$, for each $t \in T \setminus T_0$. ■

Model 1 and Model 2 represent two possible generalizations of a model proposed by Dubey and Shubik (1978) in which one commodity plays the role of money. It is straightforward to show that, in bilateral oligopoly, this model reduces to Model 2 once we label one of the two commodities as money.

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